Oscillation criteria for nonlinear neutral hyperbolic equations with functional arguments

Y. SHOUKAKU [1], I. P. STAVROULAKIS [2] AND N. YOSHIDA [3]

Abstract

This paper is devoted to the study of oscillatory behavior of solutions to nonlinear neutral hyperbolic equations with functional arguments by using the integral averaging method and generalized Riccati techniques. First, we establish oscillation results for nonlinear neutral hyperbolic equations by reducing the multi-dimensional oscillation problems to one-dimensional oscillation problems for functional differential inequalities. Secondly, we present oscillation results for nonlinear neutral hyperbolic equations by utilizing Riccati techniques.

Keywords : Oscillation, hyperbolic equations, neutral type, Riccati inequality
2000MSC : 34K11, 35B05, 35R10

1. Introduction

Consider the hyperbolic equation with functional arguments

$$\text{(E)}\quad \frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}h_i(t)u(x,\rho_i(t))\right)\right)$$
$$-a(t)\Delta u(x,t)-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}q_i(x,t)\varphi_i(u(x,\sigma_i(t)))=0,\ \ (x,t)\in\Omega\equiv G\times(0,\infty),$$

[1]Faculty of Engineering, Kanazawa University,Kanazawa 920-1192, Japan; e-mail: shoukaku@t.kanazawa-u.ac.jp
[2]Department of Mathematics, University of Ioannina 451-10, Ioannina, Greece; e-mail: ipstav@uoi.gr
[3]Department of Mathematics, University of Toyama, Toyama 930-8555, Japan; e-mail: nori@sci.u-toyama.ac.jp

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary ∂G, and the following Dirichlet and Robin (cf. [10]) boundary conditions:

(B1) $$u = 0 \quad \text{on} \quad \partial G \times [0, \infty),$$

(B2) $$\frac{\partial u}{\partial \nu} + \mu u = 0 \quad \text{on} \quad \partial G \times [0, \infty),$$

where ν denotes the unit exterior normal vector to ∂G and $\mu \in C(\partial G \times [0, \infty); [0, \infty))$.

Throughout this paper we assume that:
A1)]$r(t) \in C^1([0, \infty); (0, \infty))$,
$h_i(t) \in C^2([0, \infty); [0, \infty))$ $(i = 1, 2, \ldots, l)$,
$a(t)$, $b_i(t) \in C([0, \infty); [0, \infty))$ $(i = 1, 2, \ldots, k)$,
$q_i(x, t) \in C(\overline{\Omega}; [0, \infty))$ $(i = 1, 2, \ldots, m)$; $\rho_i(t) \in C^2([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \rho_i(t) = \infty$ $(i = 1, 2, \ldots, l)$,
$\tau_i(t) \in C([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \tau_i(t) = \infty$ $(i = 1, 2, \ldots, k)$,
$\sigma_i(t) \in C([0, \infty); \mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i = 1, 2, \ldots, m)$; $\varphi_i(s) \in C^1(\mathbb{R}; \mathbb{R})$ $(i = 1, 2, \ldots, m)$ are convex in $(0, \infty)$ and $\varphi_i(-s) = -\varphi_i(s)$ for $s \geq 0$.

Definition 1. By a *solution* of Eq. (E) we mean a function $u \in C^2(\overline{G} \times [t_{-1}, \infty)) \cap C(\overline{G} \times [\tilde{t}_{-1}, \infty))$ which satisfies (E), where

$$t_{-1} = \min\left\{0, \min_{1\leq i\leq l}\left\{\inf_{t\geq 0} \rho_i(t)\right\}, \min_{1\leq i\leq k}\left\{\inf_{t\geq 0} \tau_i(t)\right\}\right\},$$
$$\tilde{t}_{-1} = \min\left\{0, \min_{1\leq i\leq m}\left\{\inf_{t\geq 0} \sigma_i(t)\right\}\right\}.$$

Definition 2. A solution u of Eq. (E) is said to be *oscillatory* in Ω if u has a zero in $G \times (t, \infty)$ for any $t > 0$.
Definition 3. We say that the functions (H_1, H_2) belong to a function class $\mathcal{H}$, denoted by $(H_1, H_2) \in \mathcal{H}$, if $(H_1, H_2) \in C(D; [0, \infty))$ satisfy

$$H_i(t, t) = 0, \ H_i(t, s) > 0 \ (i = 1, 2) \quad \text{for } t > s,$$

where $D = \{(t, s) : 0 < s \leq t < \infty\}$, and the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on D such that

$$\frac{\partial H_1}{\partial t}(s, t) = h_1(s, t)H_1(s, t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t, s) = -h_2(t, s)H_2(t, s),$$

for some functions $h_1,\ h_2 \in C_{loc}(D;\mathbb{R})$, where $C_{loc}(D;\mathbb{R})$ denotes the set of all locally continuous functions on D.

In recent years there has been much research activity concerning the oscillation theory of nonlinear hyperbolic equations with functional arguments by employing Riccati techniques. Riccati techniques were used to obtain various oscillation results (cf. Mařík [9], Yoshida [15]). For example, we note that Kamenev-type oscillation criteria for hyperbolic equations have been obtained in [3,6,12,14]. On the other hand, interval oscillation criteria for second order differential equation have been investigated by many authors [1,3,5,6,8,12,13]. In particular, Wang, Meng and Liu [12,13] applied interval oscillation criteria to linear hyperbolic equations with functional arguments. Recently, Cui and Xu [1] presented oscillation criteria for hyperbolic equations which are not of neutral type. It seems that there are no known oscillation results for hyperbolic equations of neutral type, which are obtained by Riccati techniques.

The objective of this paper is to establish oscillation ceireria for the nonlinear neutral hyperbolic equation with functional arguments (E) by employing the Riccati method.

In Section 2 we reduce our problems to one-dimensional problems for functional differential inequalities, and second order functional differential inequalities are investigated in Section 3 via Riccati inequalities. We present oscillation results for (E) in Section 4 by combining the results of Sections 2 and 3. Two examples which illustrate our main theorems are given in Section 5.

2. Reduction to one-dimensional problems

In this section we reduce the multi-dimensional oscillation problems for (E) to one-dimensional oscillation problems. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G. Now we let

$$q_i(t) = \min_{x \in G} q_i(x,t).$$

With each solution $u(x,t)$ of the problem (E), (B1) or (E), (B2) we associate functions $U(t)$ and $\tilde{U}(t)$ respectively, defined by

$$U(t) = K_\Phi \int_G u(x,t)\Phi(x)dx,$$
$$\tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx,$$

where $K_\Phi = (\int_G \Phi(x)dx)^{-1}$ and $|G| = \int_G dx$.

Theorem 1. *If the functional differential inequality*

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(y(t)+\sum_{i=1}^{l} h_i(t)y(\rho_i(t))\right)\right)+\sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \le 0 \tag{1}$$

has no eventually positive solutions, then every solution $u(x,t)$ of the problem (E), (B1) is oscillatory in Ω.

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (E), (B1). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. (The case where $u(x,t) < 0$ can be treated similarly). Since (A2) holds, we see that $u(x,\rho_i(t)) > 0$ $(i = 1,2,\dots,l)$, $u(x,\tau_i(t)) > 0$ $(i = 1,2,\dots,k)$ and $u(x,\sigma_i(t)) > 0$ $(i = 1,2,\dots,m)$ in $G \times [t_1,\infty)$ for some $t_1 \ge t_0$. Multiplying (E) by $K_\Phi\Phi(x)$ and integrating over G, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(U(t)+\sum_{i=1}^{l} h_i(t)U(\rho_i(t))\right)\right)\\
&-a(t)K_\Phi\int_G \Delta u(x,t)\Phi(x)dx-\sum_{i=1}^{m} b_i(t)K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)dx\\
&+\sum_{i=1}^{m} K_\Phi\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx = 0,\ t\ge t_1.
\end{aligned}\tag{2}$$

From Green's formula it follows that

$$K_\Phi\int_G \Delta u(x,t)\Phi(x)dx = -\lambda_1 U(t) \le 0,\ t \ge t_1, \tag{3}$$

$$K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)dx = -\lambda_1 U(\tau_i(t)) \le 0,\ t \ge t_1. \tag{4}$$

Using the Jensen's inequality we observe that

$$\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))),\ t \geq t_1, \quad (5)$$

and combining (2)–(5), it follows that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(U(t)+\sum_{i=1}^{l} h_i(t)U(\rho_i(t))\right)\right) + \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))) \leq 0,\ t \geq t_1.$$

Therefore $U(t)$ is an eventually positive solution of (1). This is a contradiction and the proof is complete.

Theorem 2. *If the functional differential inequality (1) has no eventually positive solutions, then every solution $u(x,t)$ of the problem (E), (B2) is oscillatory in Ω.*

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (E), (B2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (A2) holds, we see that $u(x,\rho_i(t)) > 0$ $(i = 1,2,\ldots,l)$, $u(x,\tau_i(t)) > 0$ $(i = 1,2,\ldots,k)$ and $u(x,\sigma_i(t)) > 0$ $(i = 1,2,\ldots,m)$ in $G \times [t_1,\infty)$ for some $t_1 \geq t_0$. Dividing (E) by $|G|$ and integrating over G, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(\tilde{U}(t)+\sum_{i=1}^{l} h_i(t)\tilde{U}(\rho_i(t))\right)\right)\\
&-\frac{a(t)}{|G|}\int_G \Delta u(x,t)dx - \sum_{i=1}^{k}\frac{b_i(t)}{|G|}\int_G \Delta u(x,\tau_i(t))dx\\
&+\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx = 0,\ t \geq t_1. \qquad (6)
\end{aligned}$$

It follows from Green's formula that

$$\begin{aligned}
\int_G \Delta u(x,t)dx &= \int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)dS \\
&= -\int_{\partial G} \mu(x,t)u(x,t)dS \leq 0, \ t \geq t_1, \qquad (7)\\
\int_G \Delta u(x,\tau_i(t))dx &= \int_{\partial G} \frac{\partial u}{\partial \nu}(x,\tau_i(t))dS \\
&= -\int_{\partial G} \mu(x,\tau_i(t))u(x,\tau_i(t))dS \leq 0, \ t \geq t_1. \qquad (8)
\end{aligned}$$

Using the Jensen's inequality, we observe that

$$\sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))), \ t \geq t_1, \qquad (9)$$

and combining (6)–(9), it follows that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\tilde{U}(\rho_i(t))\right)\right) + \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))) \leq 0, \ t \geq t_1.$$

Therefore $\tilde{U}(t)$ is an eventually positive solution of (1). This is a contradiction and the proof is complete.

3. Second order functional differential inequalities

In this section we establish sufficient conditions for every solution $y(t)$ of the functional differential inequality (1) to have no eventually positive solution. We assume the following hypotheses:

(A1)(A2)(A3) For some $j \in \{1, 2, \ldots, m\}$, there exists a positive constants σ such that

$$\sigma_j'(t) \geq \sigma \quad \text{and} \quad \sigma_j(t) \leq t,$$

and $\varphi_j(s) \in C^1((0,\infty);(0,\infty))$, $\varphi_j'(s) > 0$ and $\varphi_j'(s)$ is nondecreasing for $s > 0$;

$$\int_{t_0}^{\infty} \frac{1}{r(t)}dt = \infty;$$

$\sum_{i=1}^{l} h_i(t) \leq 1;$

$\rho_i(t) \leq t \ (i = 1, 2, \ldots, l);$

Theorem 3. *Assume that the hypotheses (A4)–(A7) hold, and moreover assume that*

$\varphi_j(s_1 s_2) \geq \varphi_{j1}(s_1)\varphi_{j2}(s_2)$ for $s_1 \geq 0$, $s_2 > 0$, where $\varphi_{j1}(s) \in C([0,\infty); [0,\infty))$, $\varphi_{j2}(s) \in C^1((0,\infty); (0,\infty))$ and $\varphi_{j2}(s)$ is nondecreasing for $s > 0$.

If the Riccati inequality

$$z'(t) + \frac{1}{2}\frac{1}{P_{\tilde{K}}(t)} z^2(t) \leq -Q(t) \tag{10}$$

for some $\tilde{K} > 0$ and all large T, has no solution on $[T, \infty)$, where

$$P_{\tilde{K}}(t) = \frac{r(\sigma_j(t))}{2\tilde{K}\sigma}, \tag{11}$$

$$Q(t) = q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right), \tag{12}$$

then (1) has no eventually positive solutions.

Proof. Suppose that $y(t)$ is a positive solution of (1) on $[t_0, \infty)$ for some $t_0 > 0$. From (1), there exists a $j \in \{1, 2, \ldots, m\}$ such that

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(y(t) + \sum_{i=1}^{l} h_i(t)y(\rho_i(t))\right)\right) + q_j(t)\varphi_j(y(\sigma_j(t))) \leq 0, \ t \geq t_0.$$

If we define the function

$$z(t) = y(t) + \sum_{i=1}^{l} h_i(t)y(\rho_i(t)), \tag{13}$$

then we see that

$$(r(t)z'(t))' \leq -q_j(t)\varphi_j(y(\sigma_j(t))) \leq 0, \ t \geq t_0. \tag{14}$$

Since $(r(t)z'(t))' \leq 0$, $z(t) > 0$ eventually, we observe, using the hypothesis (A5), that $z'(t) \geq 0$ $(t \geq t_1)$ for some $t_1 > t_0$ (cf. [13, Lemma 2.2]). Hence $r(t)z'(t)$ is nonincreasing. Then, we find that $z'(t) \geq 0$ or $z'(t) < 0$ for

$t \geq t_1 > t_0$. First we assume that $z'(t) < 0$ for $t \geq t_1$. From the well known argument (cf. [13]) we prove that $z'(t) \geq 0$ for $t \geq t_1$. Taking into account (A6) and (A7), from (13) we see that (cf. Yoshida [15])

$$y(t) \geq \left(1 - \sum_{i=1}^{l} h_i(t)\right) z(t), \ t \geq t_1. \tag{15}$$

In view of (14) and (15), we observe that

$$(r(t)z'(t))' + q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right)\varphi_{j2}(z(\sigma_j(t))) \leq 0, \ t \geq t_1.$$

Setting

$$w(t) = \frac{r(t)z'(t)}{\varphi_{j2}(z(\sigma_j(t)))},$$

we show that

$$w'(t) = \frac{(r(t)z'(t))'}{\varphi_{j2}(z(\sigma_j(t)))} - r(t)z'(t)\frac{\varphi'_{j2}(z(\sigma_j(t)))z'(\sigma_j(t))\sigma'_j(t)}{\varphi^2_{j2}(z(\sigma_j(t)))}. \tag{16}$$

Since $z(t) > 0$, $z'(t) \geq 0$ eventually, it follows that $z(\sigma_j(t)) \geq k_0$ for some $k_0 > 0$. Hence we observe that

$$\varphi'_{j2}(z(\sigma_j(t))) \geq \varphi'_{j2}(k_0) \equiv \tilde{K}. \tag{17}$$

Substituting (17) into (16), we get

$$\begin{aligned} w'(t) \ \leq \ & -q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right) \\ & -\tilde{K}\sigma r(t)z'(t)\frac{z'(\sigma_j(t))}{\varphi^2_{j2}(z(\sigma_j(t)))}, \ t \geq t_1. \end{aligned}$$

On the other hand, (14) implies that

$$r(\sigma_j(t))z'(\sigma_j(t)) \geq r(t)z'(t),$$

and hence

$$w'(t) + \frac{1}{2}\left(\frac{2\tilde{K}\sigma}{r(\sigma_j(t))}\right)w^2(t) \leq -q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_j(t))\right) \tag{18}$$

for $t \geq t_1$. That is, $w(t)$ is a solution of (10) on $[t_1, \infty)$. This is a contradiction and the proof is complete.

Theorem 4. *Assume that the hypotheses (A4)–(A8) hold. If for each* $T > 0$ *and some* $\tilde{K} > 0$*, there exist* $(H_1, H_2) \in \mathcal{H}$*,* $\psi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \leq a < c < b$ *and*

$$\frac{1}{H_1(c,a)} \int_a^c H_1(s,a) \left\{ Q(s) - \frac{1}{4} \frac{r(\sigma_j(s))}{\tilde{K}\sigma} \lambda_1^2(s,a) \right\} \psi(s) ds + \frac{1}{H_2(b,c)} \int_c^b H_2(b,s) \left\{ Q(s) - \frac{1}{4} \frac{r(\sigma_j(s))}{\tilde{K}\sigma} \lambda_2^2(b,s) \right\} \psi(s) ds > 0, \quad (19)$$

where

$$\lambda_1(s,t) = \frac{\psi'(s)}{\psi(s)} + h_1(s,t),$$
$$\lambda_2(t,s) = \frac{\psi'(s)}{\psi(s)} - h_2(t,s).$$

Then (1) has no eventually positive solutions.

Proof. Suppose that $y(t)$ is a positive solution of (1) on $[t_0, \infty)$ for some $t_0 > 0$. At first, we assume that $y(t) > 0$ on (a, b). Proceeding as in the proof of Theorem 3, we see that there exists a function $w(s)$ which satisfies

$$Q(s)\psi(s) \leq -w'(s)\psi(s) - \frac{\tilde{K}\sigma}{r(\sigma_j(s))} w^2(s)\psi(s). \quad (20)$$

Multiplying (20) by $H_2(t,s)$ and integrating over $[c,t]$ for $t \in [c,b)$, we have

$$\begin{aligned} &\int_c^t H_2(t,s)Q(s)\psi(s)ds \\ \leq & -\int_c^t H_2(t,s)w'(s)\psi(s)ds - \int_c^t H_2(t,s)\frac{\tilde{K}\sigma}{r(\sigma_j(s))} w^2(s)\psi(s)ds \\ \leq & \ H_2(t,c)w(c)\psi(c) + \frac{1}{4}\int_c^t H_2(t,s)\lambda_2^2(t,s)\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\psi(s)ds \\ & -\int_c^t H_2(t,s)\left\{ \sqrt{\frac{\tilde{K}\sigma}{r(\sigma_j(s))}} w(s) - \frac{1}{2}\lambda_2(t,s)\sqrt{\frac{r(\sigma_j(s))}{\tilde{K}\sigma}} \right\}^2 \psi(s)ds, \end{aligned}$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(t,s)\right\}\psi(s)ds \le w(c)\psi(c).$$

Letting $t \to b^-$ in the last inequality, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}\psi(s)ds \le w(c)\psi(c). \quad (21)$$

On the other hand, multiplying (20) by $H_1(s,t)$ and integrating over $[t,c]$ for $t \in (a,c]$, we obtain

$$\begin{aligned}
&\int_t^c H_1(s,t)q_j(s)\psi(s)ds \\
\le\ & -\int_t^c H_1(s,t)w'(s)\psi(s)ds - \int_t^c H_1(s,t)\frac{\tilde{K}\sigma}{r(\sigma_j(s))}w^2(s)\psi(s)ds \\
\le\ & -H_1(c,t)w(c)\psi(c) + \frac{1}{4}\int_t^c H_1(s,t)\lambda_1^2(s,t)\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\psi(s)ds \\
& -\int_t^c H_1(s,t)\left\{\sqrt{\frac{\tilde{K}\sigma}{r(\sigma_j(s))}}w(s)-\frac{1}{2}\lambda_1(s,t)\sqrt{\frac{r(\sigma_j(s))}{\tilde{K}\sigma}}\right\}^2\psi(s)ds,
\end{aligned}$$

and therefore

$$\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,t)\right\}\psi(s)ds \le -w(c)\psi(c).$$

Letting $t \to a^+$ in the last inequality, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}\psi(s)ds \le -w(c)\psi(c). \quad (22)$$

Adding (21) and (22), we obtain the following

$$\begin{aligned}
&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}ds \\
&\quad+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{Q(s)-\frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}ds \le 0,
\end{aligned}$$

which contradicts the condition (19). Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumptions, for each $i \in \mathbb{N}$, there exists $a_i,\ b_i,\ c_i \in [0, \infty)$ such that $T_i \le a_i < c_i < b_i$, and (19) holds with $a,\ b,\ c$ replaced by $a_i,\ b_i,\ c_i$, respectively. Therefore, every nontrivial solution $y(t)$ of (1) has at least one zero $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \ge T_i,\ i \in \mathbb{N}$, we see that $y(t)$ is an oscillatory solution of (1). This is a contradiction and the proof is complete.

Theorem 5. *Assume that the hypotheses (A4)–(A8) hold. If for each $T > 0$ and some $\tilde{K} > 0$, there exist functions $(H_1, H_2) \in \mathcal{H}$, $\psi(t) \in C^1((0,\infty);(0,\infty))$, such that*

$$\limsup_{t\to\infty} \int_T^t H_1(s,T)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,T)\right\}\psi(s)ds > 0 \tag{23}$$

and

$$\limsup_{t\to\infty} \int_T^t H_2(t,s)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(t,s)\right\}\psi(s)ds > 0, \tag{24}$$

then (1) has no eventually positive solutions.

Proof. For any $T \ge t_0$, let $a = T$ and choose $T = a$ in (23). Then there exists $c > a$ such that

$$\int_a^c H_1(s,a)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_1^2(s,a)\right\}\psi(s)ds > 0. \tag{25}$$

Next, choose $T = c$ in (24). Then there exists $b > c$ such that

$$\int_c^b H_2(b,s)\left\{Q(s) - \frac{1}{4}\frac{r(\sigma_j(s))}{\tilde{K}\sigma}\lambda_2^2(b,s)\right\}\psi(s)ds > 0. \tag{26}$$

Combining (25) and (26), we obtain (19). By the virtue of Theorem 4, the proof is complete.

4. Oscillation criteria for Eq. (E)

In this section, by combining the results of Sections 2 and 3, we establish sufficient conditions for oscillation of Eq. (E).

Using the Riccati inequality, we derive sufficient conditions for every solution of hyperbolic equation (E) to be oscillatory. We are going to use the following lemma which is due to Usami [11].

Lemma. *If there exists a function* $\psi(t) \in C^1([T_0,\infty);(0,\infty))$ *such that*

$$\int_{T_1}^{\infty} \left(\frac{\bar{p}(t)|\psi'^{\beta}}{\psi(t)}\right)^{\frac{1}{\beta-1}} dt < \infty,$$
$$\int_{T_1}^{\infty} \frac{1}{\bar{p}(t)(\psi(t))^{\beta-1}} dt = \infty,$$
$$\int_{T_1}^{\infty} \psi(t)\bar{q}(t)dt = \infty$$

for some $T_1 \geq T_0$, *then the Riccati inequality*

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^{\beta} \leq -\bar{q}(t),$$

where $\beta > 1$, $\bar{p}(t) \in C([T_0,\infty);(0,\infty))$ *and* $\bar{q}(t) \in C([T_0,\infty);\mathbb{R})$, *has no solution on* $[T,\infty)$ *for all large* T.

Combining Theorems 1-3 and Lemma, we obtain the following theorem.

Theorem 6. *Assume that the hypotheses (A1)–(A7) hold. If*

$$\int_{T_1}^{\infty} \left(\frac{P_{\tilde{K}}(t)\psi'^2}{\psi(t)}\right) dt < \infty,$$
$$\int_{T_1}^{\infty} \frac{1}{P_{\tilde{K}}(t)\psi(t)} dt = \infty,$$
$$\int_{T_1}^{\infty} \psi(t)Q(t)dt = \infty,$$

where $P_{\tilde{K}}(t)$ *and* $Q(t)$ *are defined by (11) and (12) for some* $\tilde{K} > 0$, *then every solution* $u(x,t)$ *of (E), (B1) (or (E), (B2)) is oscillatory in* Ω.

Combining Theorems 1–2 and 4, we have the following theorem.

Theorem 7. *Assume that the hypotheses (A1)–(A7) hold. If for each* $T > 0$ *and some* $\tilde{K} > 0$, *there exist functions* $(H_1, H_2) \in \mathcal{H}$, $\psi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \leq a < c < b$ *and (19) hold, then every solution* $u(x,t)$ *of (E), (B1) (or (E), (B2)) is oscillatory in* Ω.

Analogously, combining Theorems 1–2 and 5 we derive the following.

Theorem 8. *Assume that the hypotheses (A1)–(A7) hold. If for each* $T > 0$ *and some* $\tilde{K} > 0$, *there exist functions* $(H_1, H_2) \in \mathcal{H}$, $\psi(t) \in$

$C^1((0,\infty);(0,\infty))$ such that (23) and (24) hold, then every solution $u(x,t)$ of (E), (B1) (or (E), (B2)) is oscillatory in Ω.

5. Examples

We present the following examples which illustrate the applicability of our results.

Example 1. Consider the problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left(e^{-t}\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\pi)\right)\right)-\frac{1}{2}e^{-t}\Delta u(x,t)\\ &\quad -\frac{1}{2}e^{-t}\Delta u\left(x,t+\frac{\pi}{2}\right)-e^{2t}\Delta u(x,t-2\pi)\\ &\quad +e^{2t}u(x,t-\pi)=0,\ (x,t)\in(0,\pi)\times[1,\infty), \qquad (27)\\ &u(0,t)=u(\pi,t)=0. \qquad (28)\end{aligned}$$

Here $n=1$, $k=2$, $m=1$, $r(t)=e^{-t}$, $h_1(t)=1/2$, $q_1(x,t)=e^{2t}$, $\sigma_1(t)=t-\pi$ and $\varphi'_{12}(\xi)=1=\tilde{K}$. It is easy to see that

$$P_{\tilde{K}}(t)=\frac{1}{2}e^{-t+\pi},\quad Q(t)=\frac{1}{2}e^{2t}.$$

By choosing

$$\psi(t)=e^{-2t},\quad H_1(s,t)=H_2(t,s)=\left(e^t-e^s\right)^2,$$

we see that

$$\begin{aligned}&\int^{\infty}\left(\frac{\frac{1}{2}e^{-t+\pi}(-2e^{-2t})^2}{e^{-2t}}\right)dt=\int^{\infty}2e^{-3t+\pi}dt<\infty,\\ &\int^{\infty}\left(\frac{1}{\frac{1}{2}e^{-t+\pi}\times e^{-2t}}\right)dt=\int^{\infty}2e^{3t-\pi}dt=\infty,\\ &\int^{\infty}\left(e^{-2t}\times\frac{1}{2}e^{2t}\right)dt=\infty.\end{aligned}$$

Choose now $a=0$, $b=2\pi$ and $c=\pi$ and observe that

$$\begin{aligned}&\frac{1}{(1-e^{\pi})^2}\int_0^{\pi}(1-e^s)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{2s}}{(e^s-1)^2}\right\}e^{-2s}ds\\ &+\frac{1}{(e^{2\pi}-e^{\pi})^2}\int_{\pi}^{2\pi}(e^{2\pi}-e^s)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{4\pi}}{(e^{2\pi}-e^s)^2}\right\}e^{-2s}ds>0,\end{aligned}$$

that is, the condition (19) is satisfied. Also

$$\begin{aligned}&\limsup_{t\to\infty}\int_T^t (e^s-s^T)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{2s}}{(e^s-e^T)^2}\right\}e^{-2s}ds\\ =\;&\limsup_{t\to\infty}\left\{\frac{1}{4}e^{2t}-e^{t+T}+\frac{1}{2}\left(t-T+\frac{3}{2}\right)e^{2T}+e^{-t+\pi}-e^{-T+\pi}\right\}>0\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_T^t (e^t-s^s)^2\left\{\frac{1}{2}e^{2s}-\frac{1}{4}e^{-s+\pi}\frac{4e^{2t}}{(e^t-e^s)^2}\right\}e^{-2s}ds\\ =\;&\limsup_{t\to\infty}\left\{\left(\frac{1}{2}\left(t-T-\frac{3}{2}\right)-\frac{1}{3}e^{\pi-3T}\right)e^{2t}+e^{t+T}+\frac{1}{3}e^{-t+\pi}-\frac{1}{4}e^{2T}\right\}>0.\end{aligned}$$

that is, the conditions (23) and (24) hold. Thus, all the conditions of Theorems 6–8 are satisfied. Therefore every solution $u(x,t)$ of the problem (27), (28) is oscillatory in $(0,\infty)\times[1,\infty)$. For example, $u(x,t)=\sin x\sin t$ is such a solution.

Example 2. Consider the problem

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{(t+\pi)^2}\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-2\pi)\right)\right)-\Delta u(x,t)\\ &\quad-\frac{3}{2(t+\pi)^2}\Delta u(x,t-2\pi)-\frac{3}{(t+\pi)^3}\Delta u\left(x,t+\frac{\pi}{2}\right)\\ &\quad+u(x,t-\pi)=0,\ (x,t)\in(0,\pi)\times[1,\infty), \qquad (29)\\ &-u_x(0,t)=u_x(\pi,t)=0. \qquad (30)\end{aligned}$$

Here $n=1$, $k=2$, $m=1$, $r(t)=(t+\pi)^{-2}$, $h_1(t)=1/2$, $q_1(x,t)=1$, $\sigma_1(t)=t-2\pi$ and $\varphi'_{12}(\xi)=1=\tilde{K}$. It is easy to see that

$$P_{\tilde{K}}(t)=\frac{1}{2t^2},\quad Q(t)=\frac{1}{2}.$$

If we choose $\psi(t)=t^2$, then

$$\begin{aligned}&\int^\infty\left(\frac{\frac{1}{2t^2}(2t)^2}{t^2}\right)dt=\int^\infty\frac{2}{t^2}dt<\infty,\\ &\int^\infty\left(\frac{1}{\left(\frac{1}{2t^2}\right)\times t^2}\right)dt=\infty,\\ &\int^\infty\left(\frac{1}{2}t^2\right)dt=\infty.\end{aligned}$$

Next, choose $\psi(t) = 1$, $H_1(s,t) = H_2(t,s) = (t-s)^2$, and $a = 0$, $b = 2\pi$, $c = \pi$. It is easy to see that

$$\frac{1}{\pi^2}\int_0^{\pi} s^2 \left\{\frac{1}{2} - \frac{1}{4s^2}\frac{4}{s^2}\right\} s^2 ds$$
$$+\frac{1}{\pi^2}\int_{\pi}^{2\pi} (2\pi - s)^2 \left\{\frac{1}{2} - \frac{1}{4s^2}\frac{4}{(2\pi - s)^2}\right\} s^2 ds > 0.$$

Moreover,

$$\limsup_{t\to\infty} \int_T^t (s-T)^2 \left\{\frac{1}{2} - \frac{1}{4}\frac{1}{s^2}\frac{4}{(s-T)^2}\right\} s^2 ds$$
$$= \limsup_{t\to\infty} \left\{\frac{1}{10}t^5 - \frac{1}{4}Tt^4 + \frac{1}{6}T^2t^3 - t - \frac{1}{60}T^5 + T\right\} > 0$$

and

$$\limsup_{t\to\infty} \int_T^t (t-s)^2 \left\{\frac{1}{2} - \frac{1}{4}\frac{1}{s^2}\frac{4}{(t-s)^2}\right\} s^2 ds$$
$$= \limsup_{t\to\infty} \left\{\frac{1}{60}t^5 - \frac{1}{6}T^3t^2 + \left(\frac{1}{4}T^4 - 1\right) t - \frac{1}{10}T^5 + T\right\} > 0.$$

Thus, all the conditions of Theorems 6-8 are satisfied. Therefore, every solution $u(x,t)$ of the problem (29), (30) is oscillatory in $(0,\pi) \times [1,\infty)$. One such solution is $u(x,t) = \cos x \sin t$.

Observe, however, that

$$\int^{\infty} \frac{1}{2}\left(\frac{3}{2(s+\pi)^2} + \frac{3}{(s+\pi)^3}\right) ds < \infty,$$

and therefore the condition (8) of Theorem 2 given by Deng [2] is not satisfied. Thus, Theorem 2 by Deng [2] can not be applied to this example.

References

[1] S. Cui, Z. Xu, Interval oscillation theorems for second order nonlinear partial delay differential equations, Differ. Equ. Appl., 1 (2009) 379–391.